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NOTES ON ENGEL GROUPS AND ENGEL ELEMENTS IN GROUPS. SOME GENERALIZATIONS

To the Memory of my Teacher, the outstanding Mathematician and Man, Professor Petr Grigorjevich Kontorovich

1. Introduction

This paper is a nostalgic reminiscence on group theory of the 1950s. In some sense this feedback to the past is inspired by the paper [1] and by the recent talks on PI-algebras given by L. Rowen and A. Kanel-Belov. Recall some definitions and some necessary old results.

We distinguish Engel groups and nil-groups, Engel elements and nil-elements, see [2].

Let $F_2 = F(x, y)$ be the free group. Define the sequence

$$e_1(x, y) = [x, y],\ e_2(x, y) = [e_1(x, y), y], \dots, e_n(x, y) = [e_{n-1}(x, y), y],$$

where $x, y \in F_2$.

Definition 1.1. An element g of a group G is called a *nil-element* if for every $a \in G$ there is $n = n(a, g)$ such that $e_n(a, g) = 1$.

Definition 1.2. A group is called a *nil-group* if every its element is a nil-element.

Every locally-nilpotent group is a nil-group, but the opposite is not true (see [3]).

Definition 1.3. A group is called *Engel* if it satisfies the identity $e_n(x, y) \equiv 1$ for some n.

In this case we call the group *n-Engel.* The variety of n-Engel groups is denoted by E_n, and let F_k^n be the free group with k free generators in this variety.

There is a long-standing conjecture that the group F_k^n is not nilpotent for some n, but up to now there are no reasonable approaches to this problem. For $n = 4$ the group F_k^4 is nilpotent, see [4]. We show (Theorem 1) that the solution of the restricted version of this problem is similar to the solution of the restricted Burnside problem (see, for example, [5, 6]).

Definition 1.4. An element g of a group G is called an *Engel element* if there exists $n = n(g)$ such that for every $x \in G$ the identity $e_n(x, g) \equiv 1$ holds in G.

Thus, the definition of an Engel element differs from the definition of a nil-element. However, following a tradition we sometimes use the term Engel element also for nil-elements (meaning unbounded Engel elements).

When do the set of all nil-elements or/and the set of all Engel elements of a group G constitute a subgroup in G? With respect to this problem, we mention the following general result [7]:

Let a group G have an ascending normal series with locally Noetherian quotients. Then the set of all its nil-elements constitutes a subgroup in G coinciding with the locally nilpotent radical $HP(G)$.

This theorem has been preceded by a similar theorem for the case when the quotients of the normal series are locally nilpotent [8] and Baer's theorem [9] stating that the nilpotent radical of a Noetherian group coincides with the set of its nil-elements.

Baer's theorem follows from the next lemma from [7].

Let G be an arbitrary group, g its nil-element. Then G has the following normal series of nilpotent subgroups

$$H_1 \subset H_2 \dots, H_n \subset \dots, \tag{1}$$

where $H_1 = \langle g \rangle$, $H_n = \langle H_{n-1}, h_n g h_n^{-1} \rangle$ for some $h_n \in G$.

Here and elsewhere $\langle\ \rangle$ stands for the subgroup generated by some elements. The series (1) stops at some place n if H_n is a normal subgroup in G.

We notice here the following result of A. Tokarenko [10], see also [2].

Let G be a subgroup in some $GL_n(K)$, where K is a commutative ring with 1. Then the set of all nil-elements in G is the locally nilpotent radical $HP(G)$.

Now we formulate the first two theorems of this paper.

Theorem 1. *In any variety E_n all its locally nilpotent groups form a subvariety.*

In the second theorem we consider PI-groups. A group is said to be a *PI-group* if it can be embedded into the group of invertible elements of some PI-algebra over a field. For example, the full matrix group $\mathrm{GL}_n(P)$ over a field P and all its subgroups are PI-groups. It has been proved by Procesi [11] and Tokarenko [12] that every periodic PI-group is locally finite. The following theorem has a similar flavor.

Theorem 2. *Every nil-PI-group G is locally nilpotent.*

In fact we will prove the following general result (see also [13]):

Theorem 3. *In every PI-group G the set of all its nil-elements coincides with the locally nilpotent radical $HP(G)$.*

Other results will require additional definitions, and they will be formulated later.

Engel groups and Engel elements in groups are related to nilpotent groups and nilpotent radicals in groups. Along with these elements we consider also their Engel-like generalizations, which are linked to solvable groups and solvable radicals in groups.

We note also two facts which will be used in the sequel. The first one is a theorem by Wilson [14] which states that every residually finite finitely generated Engel group is nilpotent. Second, we use the following theorem of Kaluzhnin [15].

Let a group G act unitriangularly and faithfully on a space V. Then the group G is nilpotent.

Unitriangularity means that there is a series

$$V = V_0 \supset V_1 \supset \ldots \supset V_n = 0$$

of subspaces in V such that all its members are invariant with respect to the action of the group G and the group G acts trivially on all quotients of the series. If the group G acts on V faithfully, then G has nilpotency class $n-1$.

2. Proof of Theorem 1

We first prove the following

Proposition 2.1. *In the variety E_n, for any natural k there exists a nilpotent group with k generators $\widetilde{F}_k^n$ such that every k-generated nilpotent group $G \in E_n$ is a homomorphic image of $\widetilde{F}_k^n$.*

Proof. Let us start with the E_n-free group F_k^n. Let H_k^n be the intersection of all normal subgroups $H \lhd F_k^n$ with nilpotent quotients F_k^n/H. The group $\widetilde{F}_k^n = F_k^n/H_k^n$ is residually nilpotent. This group is also residually finite because every finitely generated nilpotent group is residually finite. Besides, this group is Engel. By the result from [14] the group $\widetilde{F}_k^n$ is nilpotent.

Let G be an arbitrary group in E_n with k generators. There is a surjection $F_k^n \to G$. Let H be the kernel of this homomorphism. Then $H \supset H_k^n$ and this gives a surjection $\widetilde{F}_k^n \to G$.

In fact, this proposition is equivalent to Theorem 1. Indeed, let Θ be the class of all locally nilpotent groups in E_n. This class is closed with respect to taking subgroups and homomorphic images. Let us check that the class is closed under forming Cartesian products.

Let $G = \prod_{\alpha \in I} G_\alpha$, where all G_α are locally nilpotent groups in the variety E_n. Take a finitely generated subgroup H in G with k generators. The group H is approximated by k-generated nilpotent groups $H_\alpha \subset G_\alpha$. The nilpotency class of all H_α is bounded by the nilpotency class of the group $\widetilde{F}_k^n$. Thus, H is nilpotent and G is locally nilpotent.

This means that the class Θ is a variety. It is easy to see that the group $\widetilde{F}_k^n$ is the free group with k generators in the variety Θ.

3. PI-groups

Let us fix a field P and, for every group G, consider a representation $G \to A$, where A is an associative algebra with 1 over P and the arrow means a homomorphism of G to the group of invertible elements of the algebra A. If this representation is faithful, then we say that the algebra A is a *linear envelope* of the group G. The group algebra PG is the universal linear envelope. We consider groups from the point of view of their possible linear envelopes. In particular, G is a linear group if it has a finite dimensional linear envelope.

Definition 3.1. A group G is a *PI-group* if it has a linear envelope A which is a PI-algebra.

Let us fix such an algebra A and let G_0 be the group of invertible elements of A. We consider G as a subgroup in G_0.

In a PI-algebra A we consider a series of ideals

$$U_0 = 0 \subset U_1 \subset U_2 \subset A, \tag{2}$$

where U_1 is the sum of the nilpotent ideals of A and U_2 is the Levitzky radical of A. It is known [16] that U_2/U_1 is nilpotent and there is an embedding $A/U_2 \to M_n(K)$. Here $M_n(K)$ is the matrix algebra of dimension n and K is a commutative ring with 1 which is a Cartesian sum of fields. The group of invertible elements of $M_n(K)$ is $\mathrm{GL}_n(K)$.

Proposition 3.1. *Let G be a PI-group. Then there is a chain of normal subgroups*

$$1 = H_0 \subset H_1 \subset H_2 \subset G,$$

where H_1 is generated by the nilpotent normal subgroups in G, H_2 is locally nilpotent, and there is an embedding

$$G/H_2 \to GL_n(K),$$

where $K = \sum_\alpha P_\alpha$, P_α is a field.

Proof. First we recall a few known things. Let A be an associative algebra with 1, G_0 the group of invertible elements in A, G a subgroup in G_0. The group G acts in the space A by the rule: $a \to ag$, $a \in A$, $g \in G$.

Let U be a two-sided ideal in A and $\mu : A \to A/U$ the natural homomorphism. It induces the representation $\mu : G \to A/U$. Then $\mu_0(1)$ is the coset $1 + U$ and the kernel of μ_0 is the set of elements $g \in G$ such that $g - 1 \in U$, i.e. $g \in 1 + U$. We have $\mathrm{Ker}(\mu_0) = G \bigcap (1 + U)$. The group G acts also in A/U with the same kernel $G \bigcap (1 + U)$.

Consider the coset $1 + U$, and let U be a locally nilpotent ideal. We want to check that $H = 1 + U$ is a locally nilpotent normal subgroup in G_0. The set H is closed under multiplication. Let $a \in U$ and $a^n = 0$. We have

$$(1 + a)(1 - a + a^2 + \ldots + (-1)^{n-1} a^{n-1}) = 1$$

whence $1 + a$ is an invertible element. Thus $H = 1 + U$ is a subgroup in G_0. This subgroup is normal since it coincides with the kernel of the homomorphism $G_0 \to A/U$. It remains to check that H is locally nilpotent.

Consider first the case when U is a nilpotent ideal. Consider the series

$$U = U_0 \supset U_1 \supset \ldots U_k \supset \ldots \supset U_n = 0, \tag{3}$$

where U_k consists of linear combinations of elements of the form

$$a(g_1 - 1) \cdots (g_k - 1), \ a \in U, \ g_i \in H.$$

The series (3) is invariant under the action of H, and H acts trivially in the factors. Besides, H acts trivially in A/U. This means that H acts in A unitriangularly and faithfully. Thus, by Kaluzhnin's theorem, H is nilpotent.

Let now U be a locally-nilpotent ideal, $H_0 = \langle g_1, \ldots, g_n \rangle$ a finitely generated subgroup in H. Assume that for every generator g_i its inverse belongs to the set $\langle g_1, \ldots, g_n \rangle$. Then g_i, $i = 1, 2, \ldots, n$, generate H_0 as a semigroup.

For every g_i take $a_i = g_i - 1$ and generate a subalgebra U_0 in U by the elements $a_1, \ldots, a_m$. The subalgebra U_0 is nilpotent. It is easy to see that $g - 1 \in U_0$ for every $g \in H_0$ and $H_0 \subset 1 + U_0$. Take a subalgebra $U_0^* = \{U_0, 1\}$ in A. Here U_0 is a nilpotent ideal in U_0^*. The group $1 + U_0$ acts in U_0^* faithfully and unitriangularly. Hence $1 + U_0$ is nilpotent and H_0 is also nilpotent. Thus H is locally nilpotent.

Now we return to the situation when A is a PI-algebra, and let (2) be the corresponding series of ideals. Take $H_1 = G \bigcap (1 + U_1)$ which is the kernel of the action G on A/U_1. This group is locally nilpotent. In U_1 there is a directed system of nilpotent ideals U_α of the algebra A. All $G \bigcap (1 + U_\alpha) = H_\alpha$ are nilpotent normal subgroups in G and they constitute a directed system that generates H_1.

Further, take $G \bigcap (1 + U_2) = H_2$. This is a locally nilpotent normal subgroup in G which coincides with the kernel of the action G on A/U_2. The group H_1 is the kernel of action of the group H_2 on A/U_1. This action is unitriangular and H_2/H_1 is a nilpotent group.

Consider a representation $G \to A/U_2$. It corresponds to the faithful representation $G/H_2 \to A/U_2$. There is also an embedding $A/U_2 \to M_n(K)$. This induces an embedding $G/H_2 \to \mathrm{GL}_n(K)$.

Observe also the following. Let $K = \sum_\alpha P_\alpha$. For every α, take an ideal U_α with $K/U_\alpha \simeq P_\alpha$ and $\bigcap_\alpha U_\alpha = 0$. Then for every α there is a homomorphism $\mathrm{GL}_n(K) \to \mathrm{GL}_n(P_\alpha)$. Its kernel is the congruence-subgroup in $\mathrm{GL}_n(K)$ modulo the ideal U_α. This leads to the presentation of $GL_n(K)$ as a subdirect product of the groups $\mathrm{GL}_n(P_\alpha)$.

4. Theorems 2 and 3

Let us repeat the formulations of the theorems:

Every nil-PI-group is locally nilpotent (Theorem 2).

In every PI-group the set of all nil-elements coincides with the locally nilpotent radical HP(G) (Theorem 3).

Theorem 2 follows from Theorem 3. Indeed, if every element g is a nil-element then $G = HP(G)$ and G is locally nilpotent.

As for Theorem 3, it is in fact proved in [2, 10, 12, 13]. For the sake of self-completeness of the text we give here a proof of this theorem. Another reason is that the same scheme works for the proof of Theorem 6. We split the proof into 3 steps.

1. Let, first, G be a linear group, i. e. $G \subset \mathrm{GL}_n(P)$. Check that all its nilpotent elements lie in the radical $\mathrm{HP}(G)$.

Let H be a subgroup in G generated by all its nil-elements. Show that H is locally solvable. Take in H a subgroup H_0 which is generated by a finite number of nil-elements. According to a well-known theorem of A. I. Mal'cev [17], H_0 contains a system of normal subgroups T_α, $\alpha \in I$, with the trivial intersection and with finite quotients $H_\alpha = H_0/T_\alpha$. These quotients H_α are linear groups of the same dimension n over finite fields. Every H_α is generated by nil-elements whence it is nilpotent by Baer's theorem. Therefore H_α is solvable. Observe that all these H_α have solvable length bounded by a number which depends only on n. Then H_0 is also solvable and H is locally solvable. It is known that a locally solvable linear group is solvable [18]. Thus H is a solvable normal subgroup in G generated by nil-elements. According to [8] such a group is locally nilpotent. All nil-elements of the group G lie in HP-radical of G.

2. Now consider the case when $G \subset \mathrm{GL}_n(K)$. The group G is approximated by subgroups of linear groups $\mathrm{GL}_n(P_\alpha)$. As before, let H be the subgroup in G generated by all nil-elements. This H is approximated by subgroups $H_\alpha \subset \mathrm{GL}_n(P_\alpha)$. The subgroups H_α are generated by nil-elements and, hence, are solvable. The solvable lengths are bounded for all H_α. Therefore H is solvable. Since H is generated by nil-elements, H is a locally nilpotent invariant subgroup. Every nil-element lies in H, and therefore, in $\mathrm{HP}(G)$.

3. General case. We have the chain

$$H_0 = 1 \subset H_1 \subset H_2 \subset H_3 \subset G,$$

where H_1 and H_2 are the same as in Proposition 3.1, and $H_3/H_2 = HP(G/H_2)$. Let g be a nil-element in G. Take a nil-element $\overline{g} = gH_2$ in G/H_2. We have $\overline{g} \in H_3/H_2$, $g \in H_3$. Using again [8], we have $g \in HP(H_3)$. Since $HP(H_3) \subset HP(G)$, then $g \in HP(G)$.

As we have mentioned, Theorem 2 is close to the Procesi–Tokarenko theorem on periodic PI-groups. In some sense Theorem 2 is related also to the theorem from [19] where the profinite completion of a residually finite group is considered. The profinite setting also allows one to proceed from nil-groups (not from Engel groups).

5. Generalizations

Below we deal with group words of the form $u = u(x, y)$ where x and y freely generate the free group $F_2 = F(x, y)$. Consider a sequence $\overrightarrow{u} = u_1, u_2, u_3, \dots$. Such a sequence is called *correct*, if it satisfies the following two properties:

1) $u_n(a, 1) = 1$ and $u_n(1, g) = 1$ for every n, every group G and every pair of elements $a, g \in G$;

2) if $u_n(a, g) = 1$, then for every $m > n$ we have $u_m(a, g) = 1$ where $a, g \in G$.

Thus, if the identity $u_n(x, y) \equiv 1$ holds in G, then for every $m > n$ the identity $u_m(x, y) \equiv 1$ also holds in G.

For every correct sequence $\overrightarrow{u}$, consider the class of groups $\Theta = \Theta(\overrightarrow{u})$ defined by the rule: $G \in \Theta$ if there exists n such that the identity $u_n(x, y) \equiv 1$ holds in G.

For every group G, we denote by $G(\overrightarrow{u})$ the subset in G defined by the rule: $g \in G(\overrightarrow{u})$ if for every $a \in G$ there exists $n = n(a, g)$ such that $u_n(a, g) = 1$. Elements of $G(\overrightarrow{u})$ are viewed as Engel elements with respect to the given correct sequence $\overrightarrow{u}$. We call these elements $\overrightarrow{u}$*-Engel-like elements.*

If $\overrightarrow{u} = \overrightarrow{e} = e_1, \dots, e_n$ where the words $e_n(x, y)$ are defined by

$$e_1(x, y) = [x, y], \dots, e_n(x, y) = [e_{n-1}(x, y), y], \dots,$$

then $\Theta(\overrightarrow{e})$ is the class of all Engel groups. In case of finite groups the class $\Theta(\overrightarrow{e})$ coincides with the class of finite nilpotent groups.

For a finite group G, the set $G(\overrightarrow{e})$ coincides with the nilpotent radical of G.

Problem 1. *Describe $\overrightarrow{u}$ such that $\Theta(\overrightarrow{u})$ is the class of all finite solvable groups.*

Concerning this problem see [1, 14].

Problem 2. *Construct a sequence $\overrightarrow{u}$ such that, for every finite group G, the set $G(\overrightarrow{u})$ is the solvable radical of G.*

It is not known whether or not there exists such a sequence $\overrightarrow{u}$. In both the problems, an emphasis is made on the fact that we look for two variable sequences. However, similar problems can be considered also in the general case where the number of variables is not restricted. In particular, we will consider some other approach to the problem of describing the solvable radical which makes sense also for finite groups.

6. Further generalizations

For each given correct sequence $\overrightarrow{u}$, we define a new set $\overrightarrow{\overrightarrow{u}}$ as follows. Consider the free group $F = F(X, y)$, where $X = \{x_1, x_2, \dots, x_k, \dots\}$, and y is a distinguished variable. We will index words from F by the sequences of natural numbers $(n_1, n_2, \dots, n_k)$. Define the words

$$u_{(n_1,n_2,\dots,n_k)}(x_1, x_2, \dots, x_k; y)$$

by the rule: $u_{n_1}(x_1, y)$ coincides with the corresponding element of the sequence $\overrightarrow{u}$. Then, by induction,

$$u_{(n_1,n_2,\dots,n_k)}(x_1, x_2, \dots, x_k; y) = u_{n_k}(x_k; u_{(n_1,n_2,\dots,n_{k-1})}(x_1, x_2, \dots, x_{k-1}; y)).$$

We denote the set of such words (obtained by superpositions of two variable words) by $\overrightarrow{\overrightarrow{u}}$.

It is easy to see that the following associativity takes place:

$$u_{(n_1,n_2,\dots,n_k)}(x_1, x_2, \dots, x_k; y) =$$
$$= u_{(n_{l+1},\dots,n_k)}(x_{l+1}, \dots, x_k; u_{(n_1,n_2,\dots,n_l)}(x_1, x_2, \dots, x_l; y)).$$

In particular,

$$u_{(n_1,n_2,\dots,n_k)}(x_1, x_2, \dots, x_k; y) =$$
$$= u_{(n_2,\dots,n_k)}(x_2, \dots, x_k; u_{n_1}(x_1; y)).$$

Correctness of the initial sequence $\overrightarrow{u}$ induces some correctness conditions of the system $\overrightarrow{\overrightarrow{u}}$. For example, if for $l < k$ the group G satisfies the identity

$$u_{(n_l,\dots,n_k)}(x_l, \dots, x_k; y) \equiv 1$$

or the identity

$$u_{(n_1,\dots,n_{l-1})}(x_1, \dots, x_{l-1}; y) \equiv 1,$$

then G satisfies the identity

$$u_{(n_1,\dots,n_k)}(x_1, \dots, x_k; y) \equiv 1.$$

There are also other relations of such kind.

For a given system $\overrightarrow{u}$, consider the class of groups $\Theta = \Theta(\overrightarrow{u})$. By definition, a group G belongs to Θ if an identity of the form

$$u_{(n_1,\ldots,n_k)}(x_1,\ldots,x_k;y) \equiv 1$$

holds in G. From the observations above, it follows that the class Θ is a pseudovariety of groups. Besides that, for every group G we define a class of elements $G(\overrightarrow{u})$ by the rule: $g \in G(\overrightarrow{u})$ if for some $k = k(g)$ and for every sequence $(a_1,\ldots,a_k)$ of elements in G there is a set $(n_1,\ldots,n_k)$ such that

$$u_{(n_1,\ldots,n_k)}(a_1,\ldots,a_k;g) = 1$$

is fulfilled. Here the set $(n_1,\ldots,n_k)$ should be compatible with the set $(a_1,\ldots,a_k)$. This means that n_1 depends on a_1 and g, and does not depend on $(a_2,\ldots,a_k)$; n_2 depends on a_1, a_2, and g, and does not depend on $(a_3,\ldots,a_k)$, etc.; n_k depends on $a_1, a_2, \ldots, a_k$ and g.

Here we encounter a general problem of describing the sets $G(\overrightarrow{u})$ for the different sequences $\overrightarrow{u}$.

Consider a special case of $G(\overrightarrow{u})$. Let $\varepsilon = \overrightarrow{e}$ and take the sequence

$$\overrightarrow{\varepsilon} : e_1, e_2, \ldots, e_n, \ldots .$$

Consider the system $\overrightarrow{\varepsilon}$ and using this system define quasi-nil elements in groups. An element $g \in G$ is called *quasi-nil* if $g \in G(\overrightarrow{\varepsilon})$. This means that for g there exists $k = k(g)$ such that for any sequence $a_1,\ldots,a_k$, $a_i \in G$, there is a compatible set $(n_1,\ldots,n_k)$ such that

$$\varepsilon_{(n_1,\ldots,n_k)}(a_1,\ldots,a_k;g) = 1.$$

For the sequence ε, we also have the class of groups $\Theta(\overrightarrow{\varepsilon})$. Groups from this class can be considered simultaneously as generalized nilpotent and generalized solvable groups.

Denote by $E_{(n_1,\ldots,n_k)}$ the variety defined by the identity

$$\varepsilon_{(n_1,\ldots,n_k)}(x_1,\ldots,x_k;y) \equiv 1.$$

The class Θ is the union of such varieties. Each variety of the type $E_{(1,\ldots,1)}$ is nilpotent, while each variety of the type $E_{(1,2,\ldots,2)}$ contains a solvable subvariety. Besides that, any product of varieties of the type $E_{(n_1,\ldots,n_k)}$ is a subvariety in a variety of the same type. This observation applies, in particular, to the product $E_{n_1}E_{n_2}\cdots E_{n_k}$ whence this variety lies in the variety $E_{(n_k,n_{k-1}+1,\ldots,n_1+1)}$.

Return now to quasi-nil elements in groups. Let $k = k(g)$ be the minimal number such that for every $(a_1,\ldots,a_k)$ there is a compatible set $(n_1,\ldots,n_k)$ with

$$\varepsilon_{(n_1,\ldots,n_k)}(a_1,\ldots,a_k;g) = 1.$$

We call such $k = k(g)$ the *nil-order* of g. Nil-order 1 means that the element is a nil-element, nil-order 2 means that the element is not nil, but for a_1 and a_2 there are n_1, n_2 with $\varepsilon_{(n_1,n_2)}(a_1, a_2; g) = 1$. In general for $k-1$ we have some elements $(a_1^0, \ldots, a_{k-1}^0)$ such that

$$\varepsilon_{(n_1,\ldots,n_{k-1})}(a_1^0, \ldots, a_{k-1}^0; g) \neq 1$$

for an arbitrary compatible set $(n_1, \ldots, n_{k-1})$.

Let us add to $(a_1^0, \ldots, a_{k-1}^0)$ an arbitrary element a. Then for the sequence $(a_1^0, \ldots, a_{k-1}^0, a)$, there is a corresponding set $(n_1^0, \ldots, n_{k-1}^0, n)$ with the condition

$$e_n(a; g_0) = 1,$$

where

$$g_0 = \varepsilon_{(n_1^0,\ldots,n_{k-1}^0)}(a_1^0, \ldots, a_{k-1}^0; g).$$

Here the element g_0 is not trivial, the element a does not depend on g_0. The equality $e_n(a, g_0) = 1$ now means that the element g_0 is a non-trivial nil-element.

Simultaneously, we have proved the following

Proposition 6.1. *If a group G contains a non-trivial quasi-nil element g, then G contains also a non-trivial nil-element g_0.*

Now we note the next two properties related to the definition of a quasi-nil element.

1. Let H be a subgroup in G and $g \in H$ be a quasi-nil element in G. Then g is a quasi-nil element in H.

2. Let $\mu : G \to H$ be a surjection and let g be a quasi-nil element in G. Then $\mu(g)$ is a quasi-nil element in H.

Indeed, take $k = k(g)$ and the corresponding presentation

$$\varepsilon_{(n_1,\ldots,n_k)}(a_1, \ldots, a_k; g) = 1.$$

Then

$$\varepsilon_{(n_1,\ldots,n_k)}(\mu(a_1), \ldots, \mu(a_k; \mu(g)) = 1.$$

Here $\mu(a_1), \ldots, \mu(a_n)$ are arbitrary elements in H.

It is clear that along with quasi-nil elements it is quite natural to define quasi-Engel elements that generalize Engel elements.

7. Some radicals

Let G be a group. Consider in G the locally nilpotent radical $\mathrm{HP}(G) = R(G)$ and the locally noetherian radical $NR(G)$. The corresponding upper radicals will be denoted by $\widetilde{R}(G)$ and $\widetilde{NR}(G)$. These radicals are obtained by iterations of the initial $R(G)$ and $NR(G)$. Namely, consider the series (upper radical series)

$$1 = R_0 \subset R = R_1 \subset \ldots \subset R_\alpha \subset \ldots,$$

where $R_{\alpha+1}/R_\alpha$ is $R(G/R_\alpha)$. This series terminates at some $R_\gamma = \widetilde{R}(G)$. Then $\widetilde{R}(G)$ is the upper radical for the radical $R(G)$. The factor group $G/\widetilde{R}(G)$ is locally nilpotent semi-simple, i.e. it does not contain non-trivial locally nilpotent normal subgroups.

The radical $\widetilde{R}$ is defined also by the class of groups G which has ascending normal series with locally nilpotent factors. Such groups are called *radical*, see [8]. In finite groups the radical $\widetilde{R}(G)$ coincides with the solvable radical.

The radical $\widetilde{NR}(G)$ is defined following the same scheme as for the radical $\widetilde{R}(G)$. If $\widetilde{NR}(G) = G$, the group G is called *noetherian radical group.*

8. Theorems on radical characterization

Let us take in the upper radical series of a group G the members with finite indices

$$1 = R_0 \subset R_1 \subset \ldots R_k \subset \ldots .$$

Proposition 8.1. *An element g which belongs to R_k for some k and does not belong to R_{k-1} is a quasi-nil element of nil-order k.*

Proof. For the case $g \in R_1$ this is true. Further, we proceed by induction. Suppose that for $g \in R_{k-1}$ it is proved that the nil-order of this g is $\leqslant k-1$. Let $g \in R_k$. Take a sequence of elements $a_1, \ldots, a_k$ in G and for a_1 and g find n_1 with $e_{n_1}(a_1, g) \in R_{k-1}$. Apply the induction assumption to the element $e_{n_1}(a, g)$. We have

$$\varepsilon_{(n_2,\ldots,n_k)}(a_2, \ldots, a_k; e_{n_1}(a_1, g)) = 1 = \varepsilon_{(n_1,\ldots,n_k)}(a_1, \ldots, a_k; g).$$

Hence the nil-order of the element g is at most k. Prove further that it is exactly k. Let g is of the order $l \leqslant k$. Take $a_1, \ldots, a_l, n_1, \ldots, n_l$ such that

$$\varepsilon_{(n_1,\ldots,n_l)}(a_1, \ldots, a_l; g) = 1 = e_{n_l}(a_l; \varepsilon_{(n_1,\ldots,n_{l-1})}(a_1, \ldots, a_{l-1}; g).$$

The element a_l does not depend on $g_0 = \varepsilon_{(n_1,\ldots,n_{l-1})}(a_1, \ldots, a_{l-1}; g)$, and all these g_0 are nil-elements (for all $a_1, \ldots, a_{l-1}$). Some of g_0 are non-trivial and all of them lie in R_1. Consider G/R_1. Here all $\overline{g}_0$ are trivial and the nil-order of $\overline{g}$ is $\leqslant l-1$. By the induction assumption, $\overline{g} \in R_l/R_1$. Then $g \in R_l$. By the condition, g does not belong to R_{k-1}. Then $l = k$.

Proposition 8.2. *Let $\widetilde{NR}(G) = G$ and $\widetilde{R}(G)$ be the radical. Then every quasi-nil element $g \in G$ belongs to $\widetilde{R}(G)$.*

Proof. Let g be a quasi-nil element which does not belong to $\widetilde{R}(G)$. Then the element $\overline{g} = g\widetilde{R}(G)$ is quasi-nil in the semi-simple group $\overline{G} = G/\widetilde{R}(G)$. If $g \neq 1$, then there exists a non-trivial nil element in $\overline{G}$. We obtain a contradiction with the semi-simplicity of G.

Propositions 8.1 and 8.2 immediately imply

Theorem 4. *Let $\widetilde{NR}(G) = G$ and let the upper radical series in G be of finite length. Then $\widetilde{R}(G)$ coincides with the set of all quasi-nil elements in G.*

We have seen that every quasi-nil element in $\widetilde{NR}(G) = G$ lies in $\widetilde{R}(G)$ for upper radical series of any length. However, in this general situation we cannot state that every element from $\widetilde{R}(G)$ is quasi-nil. In order to include this case in the general setting we define unbounded quasi-nil elements. In this unbounded approach we do not fix $k = k(g)$, since we do not know in advance what are the length of words which are related to the given g. Thus we consider infinite sequences $\overline{a} = (a_1, a_2, \ldots, a_k, \ldots)$. We call an element *unbounded quasi-nil* if for any $\overline{a}$ there is $k = k(\overline{a}, g)$ and compatible $(n_1, n_2, \ldots, n_k)$ such that

$$\varepsilon_{(n_1,n_2,\ldots,n_k)}(a_1, a_2, \ldots, a_k; g) = 1.$$

Theorem 5. *For any group G, every element in $\widetilde{R}(G)$ is an unbounded quasi-nil element.*

Proof. We start with the upper radical series

$$1 = R_0 \subset R_1 \subset R_2 \subset \ldots \subset R_\alpha \subset \ldots \subset R_\gamma = R$$

and use induction. For $g \in R_1$, the statement is evident and let for all $\beta < \alpha$ the statement is true. Show that every $g \in R_\alpha$ is unbounded quasi-nil.

If α is a limit ordinal, then $g \in R_\beta$ with $\beta < \alpha$ and g is unbounded quasi-nil.

Now suppose that there exists $\alpha - 1$. For the given $g \in R_\alpha$, take a sequence

$$\overline{a} = (a_1, a_2, \ldots, a_k, \ldots).$$

For a_1 we find n_1 with $e_{n_1}(a_1, g) \in R_{\alpha-1}$. The element $e_{n_1}(a_1, g)$ is unbounded quasi-nil. For this element take the sequence $a_2, \ldots, a_k, \ldots$, and let $(n_2, \ldots, n_k,)$ be defined for this sequence. Here, k depends also on a_1. We have

$$\varepsilon_{(n_2,\ldots,n_k)}(a_2, \ldots, a_k; e_{n_1}(a_1, g)) = 1$$

whence

$$\varepsilon_{(n_1,n_2\ldots,n_k)}(a_1, a_2, \ldots, a_k; g) = 1.$$

Thus, the element g satisfies the condition of being unbounded quasi-nil.

9. Again about PI-groups

Theorem 3 can be applied to finite groups. It can be also applied to linear groups over fields and, as we will see soon, to any PI-groups. In these cases the conditions of the type $\widetilde{NR}(G) = G$ are not necessary.

Theorem 6. *If G is a PI-group, then its "solvable" radical $\widetilde{R}(G)$ coincides with the set of all quasi-nil elements.*

The proof of this theorem follows the scheme used in the proof of Theorem 3. The only observation has to be taken into account is the fact that in every solvable group its solvable radical coincides with the set of its nil-elements.

In particular, we can state that a PI-group is "solvable" (in the sense that $\widetilde{R}(G) = G$) if and only if all elements in G are quasi-nil.

From Theorem 2 it follows that if in a PI-group every two elements generate a nilpotent subgroup then the whole group is locally nilpotent. Now we consider the case when every two elements generate a solvable subgroup.

Theorem 7. *Let G be a PI-group and let every two elements in G generate a solvable subgroup. Then G is solvable modulo locally nilpotent radical $HP(G)$.*

Proof. It is sufficient to consider the case when G is a subgroup in a $\mathrm{GL}_n(K)$, where K is a direct sum of fields. If K is a field the proof follows from [18, 20]. The proof for the general case imitates the reduction to the field case in the previous theorem.

Remark. In every PI-group G the group $\widetilde{R}(G)/R(G)$ is solvable. From the paper [21], it follows that in every PI-group the radical $\widetilde{(G)}$ coincides with its locally solvable radical which do exist in such groups. In the paper [22] it is proved that the locally solvable radical is the set of all elements $g \in G$ with the property: for any $a \in G$ the subgroup generated by a and g is solvable.

10. Conclusion

All above can be applied to finite groups. However, the problems 1 and 2 remains open. Their solution should use some "subtle" theory of finite groups. This is the classification of finite simple groups and their automorphisms, equations in finite simple groups, etc. Here some algebraic geometry can be used. Besides that, along with Engel-like elements the corresponding Engel-like automorphisms should be considered.

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